Chapitre 7 : Calculs de déterminants

$$\Delta_1 = \begin{vmatrix} 1^2 & 2^2 & 3^2 & \cdots & (n-1)^2 & n^2 \\ 2^2 & 3^2 & 4^2 & \cdots & n^2 & (n+1)^2 \\ 3^2 & 4^2 & 5^2 & \cdots & (n+1)^2 & (n+2)^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (n-1)^2 & n^2 & (n+1)^2 & \cdots & (2n-3)^2 & (2n-2)^2 \\ n^2 & (n+1)^2 & (n+2)^2 & \cdots & (2n-2)^2 & (2n-1)^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1^2 & 2^2-1^2 & 3^2-2^2 & \cdots & (n-1)^2-(n-2)^2 & n^2-(n-1)^2 \\ 2^2 & 3^2-2^2 & 4^2-3^2 & \cdots & n^2-(n-1)^2 & (n+1)^2-n^2 \\ 3^2 & 4^2-3^2 & 5^2-4^2 & \cdots & (n+1)^2-n^2 & (n+2)^2-(n+1)^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (n-1)^2 & n^2-(n-1)^2 & (n+1)^2-n^2 & \cdots & (2n-3)^2-(2n-4)^2 & (2n-2)^2-(2n-3)^2 \\ n^2 & (n+1)^2-n^2 & (n+2)^2-(n+1)^2 & \cdots & (2n-2)^2-(2n-3)^2 & (2n-1)^2-(2n-2)^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1^2 & 3 & 5 & \cdots & (n-1)^2-(n-2)^2 & 2n-1 \\ 2^2 & 5 & 7 & \cdots & 2n-1 & 2n+1 \\ 3^2 & 7 & 9 & \cdots & 2n+1 & 2n+3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (n-1)^2 & 2n-1 & 2n+1 & \cdots & 4n-7 & 4n-5 \\ n^2 & 2n+1 & 2n+3 & \cdots & 4n-5 & 4n-3 \end{vmatrix}$$

$$= \begin{vmatrix} 1^2 & 3 & 2 & \cdots & 2 & 2 \\ 2^2 & 5 & 2 & \cdots & 2 & 2 \\ 3^2 & 7 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (n-1)^2 & 2n-1 & 2 & \cdots & 2 & 2 \\ n^2 & 2n+1 & 2 & \cdots & 2 & 2 \end{vmatrix}$$

$$\Delta_2 = D_n = \begin{vmatrix} b & a & a & \cdots & a \\ c & b & a & \ddots & \vdots \\ c & c & \ddots & \ddots & a \\ \vdots & \vdots & \ddots & b & a \\ c & c & \cdots & c & b \end{vmatrix}_n = \begin{vmatrix} b-a & 0 & 0 & \cdots & 0 & a \\ c-a & b-a & 0 & \cdots & 0 & a \\ c-a & c-a & b-a & \ddots & \vdots & a \\ \vdots & \vdots & \ddots & \ddots & 0 & a \\ c-a & c-a & \cdots & c-a & b-a & a \\ c-b & c-b & \cdots & c-b & c-b & b \end{vmatrix}_n$$

$$= (b-a)\begin{vmatrix} b-a & 0 & 0 & \cdots & 0 & a \\ c-a & b-a & 0 & \cdots & 0 & a \\ c-a & c-a & b-a & \ddots & \vdots & a \\ \vdots & \vdots & \vdots & \ddots & 0 & a \\ c-a & c-a & c-a & \cdots & b-a & a \\ c-b & c-b & c-b & \cdots & c-b & b \end{vmatrix}_{n-1} + (-1)^{n+1} a \underbrace{\begin{vmatrix} c-a & b-a & 0 & \cdots & 0 & 0 \\ c-a & c-a & b-a & \ddots & 0 & 0 \\ c-a & c-a & c-a & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & b-a & 0 \\ c-a & c-a & c-a & \cdots & c-a & b-a \\ c-b & c-b & c-b & \cdots & c-b & c-b \end{vmatrix}_{n-1}}_{\delta_{n-1}}$$

$$D_n = (b-a)D_{n-1} + (-1)^{n+1} a\,\delta_{n-1}$$

$$\delta_n = (c-a)\delta_{n-1} - (b-a)\delta_{n-1} = (c-b)\delta_{n-1} \quad \Rightarrow \quad \delta_n = (c-b)^{n-2}\delta_2 = (c-b)^{n-2}\begin{vmatrix} c-a & b-a \\ c-b & c-b \end{vmatrix} = (c-b)^n$$

$$D_n = (b-a)D_{n-1} + (-1)^{n+1} a (c-b)^{n-1}$$

$$\underset{b\neq a}{\Rightarrow} \quad \frac{D_n}{(b-a)^n} - \frac{D_{n-1}}{(b-a)^{n-1}} = \frac{a}{b-a}\left(\frac{b-c}{b-a}\right)^{n-1}$$

$$\Rightarrow \quad \sum_{k=2}^{n}\left(\frac{D_k}{(b-a)^k} - \frac{D_{k-1}}{(b-a)^{k-1}}\right) = \frac{a}{b-a}\sum_{k=2}^{n}\left(\frac{b-c}{b-a}\right)^{k-1}$$

$$\underset{c\neq a}{\Rightarrow} \quad \frac{D_n}{(b-a)^n} - \frac{D_1}{b-a} = \frac{a}{b-a}\frac{b-c}{b-a}\frac{\left(\frac{b-c}{b-a}\right)^{n-1} - 1}{\frac{b-c}{b-a} - 1}$$

$$\Rightarrow \quad \frac{D_n}{(b-a)^n} = \frac{b}{b-a} - \frac{a}{a-c}\frac{b-c}{b-a} + \frac{a}{a-c}\frac{(b-c)^n}{(b-a)^n}$$

$$\Rightarrow \quad \boxed{D_n = \frac{a(b-c)^n - c(b-a)^n}{a-c}}$$

$$\boxed{D_n \underset{c\to a}{=} \big[(n-1)a + b\big](b-a)^{n-1}}$$

$$\Delta_4 = \begin{vmatrix} a & a_2 & a_3 & \cdots & a_n \\ a_1 & a & a_3 & \cdots & a_n \\ a_1 & a_2 & a & \cdots & \vdots \\ \vdots & \vdots & \vdots & & a_n \\ a_1 & a_2 & a_3 & \cdots & a \end{vmatrix} = \det\left(a_1U + (a-a_1)E_1,\, a_2U + (a-a_2)E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$= a_1 \det\left(U,\, a_2U + (a-a_2)E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$
$$+ (a-a_1)\det\left(E_1,\, a_2U + (a-a_2)E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$= a_1 \det\left(U,\, (a-a_2)E_2,\, \ldots,\, (a-a_n)E_n\right)$$
$$+ (a-a_1)\det\left(E_1,\, a_2U + (a-a_2)E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$= a_1(a-a_2)\ldots(a-a_n)\det\left(U,\, E_2,\, \ldots,\, E_n\right)$$
$$+ (a-a_1)\det\left(E_1,\, a_2U + (a-a_2)E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$= a_1(a-a_2)(a-a_n)$$
$$+ (a-a_1)\det\left(E_1,\, a_2U + (a-a_2)E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$= a_1(a-a_2)\ldots(a-a_n)$$
$$+ a_2(a-a_1)\det\left(E_1,\, U,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$
$$+ (a-a_1)(a-a_2)\det\left(E_1,\, E_2,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$= a_1(a-a_2)\ldots(a-a_n) + a_2(a-a_1)(a-a_n)\ldots(a-a_n)$$
$$+ (a-a_1)(a-a_2)\det\left(E_1,\, E_2,\, a_3U + (a-a_3)E_3,\, \ldots,\, a_nU + (a-a_n)E_n\right)$$

$$\vdots$$

$$= a_1(a-a_2)\ldots(a-a_n) + a_2(a-a_1)(a-a_n)\ldots(a-a_n) + \ldots + a_{n-1}(a-a_1)\ldots(a-a_{n-2})(a-a_n)$$
$$+ (a-a_1)(a-a_2)\ldots(a-a_{n-1})\det\left(E_1,\, E_2,\, \ldots,\, E_{n-1},\, a_nU + (a-a_n)E_n\right)$$

$$= a_1(a-a_2)\ldots(a-a_n) + a_2(a-a_1)(a-a_n)\ldots(a-a_n) + \ldots + a_{n-1}(a-a_1)\ldots(a-a_{n-2})(a-a_n)$$
$$+ a_n(a-a_1)(a-a_2)\ldots(a-a_{n-1})\det\left(E_1,\, E_2,\, \ldots,\, E_{n-1},\, U\right)$$
$$+ (a-a_1)(a-a_2)\ldots(a-a_{n-1})(a-a_n)\det\left(E_1,\, E_2,\, \ldots,\, E_{n-1},\, E_n\right)$$

$$= a_1(a-a_2)\ldots(a-a_n) + a_2(a-a_1)(a-a_n)\ldots(a-a_n) + \ldots + a_{n-1}(a-a_1)\ldots(a-a_{n-2})(a-a_n)$$
$$+ a_n(a-a_1)(a-a_2)\ldots(a-a_{n-1})$$
$$+ (a-a_1)(a-a_2)\ldots(a-a_{n-1})(a-a_n)$$

$$\Delta\left(a_1,\dots,a_n,b_1,\dots,b_n\right)=P(b_n)$$

$$P(x)=\begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+x} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+x} \\ \vdots & \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \frac{1}{a_n+b_2} & \cdots & \frac{1}{a_1+b_n} \end{vmatrix}=\lambda\frac{\prod_{i=1}^{n-1}(x-b_i)}{\prod_{i=1}^{n}(a_i+x)}$$

$$xP(x)\xrightarrow[x\to+\infty]{}\lambda=\begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_{n-1}} & 1 \\ \vdots & & \vdots & \vdots \\ \frac{1}{a_{n-1}+b_1} & \cdots & \frac{1}{a_{n-1}+b_{n-1}} & 1 \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_{n-1}} & 1 \end{vmatrix}_n=\begin{vmatrix} \frac{a_n-a_1}{(a_1+b_1)(a_n+b_1)} & \cdots & \frac{a_n-a_1}{(a_1+b_{n-1})(a_n+b_{n-1})} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{a_n-a_{n-1}}{(a_{n-1}+b_1)(a_n+b_1)} & \cdots & \frac{a_n-a_{n-1}}{(a_{n-1}+b_{n-1})(a_n+b_{n-1})} & 0 \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_{n-1}} & 1 \end{vmatrix}_n$$

$$=\frac{(a_n-a_1)\dots(a_n-a_{n-1})}{(a_n+b_1)\dots(a_n+b_{n-1})}\begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_{n-1}} \\ \vdots & & \vdots \\ \frac{1}{a_{n-1}+b_1} & \cdots & \frac{1}{a_{n-1}+b_{n-1}} \end{vmatrix}_{n-1}=\frac{\prod_{k=1}^{n-1}(a_n-a_k)}{\prod_{k=1}^{n-1}(a_n+b_k)}\Delta\left(a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}\right)$$

$$\Delta_5=P(b_n)=\frac{\prod_{i=1}^{n-1}(b_n-b_i)}{\prod_{i=1}^{n-1}(a_i+b_n)}\frac{\prod_{k=1}^{n-1}(a_n-a_k)}{\prod_{k=1}^{n}(a_n+b_k)}\Delta\left(a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}\right)=\frac{\prod_{1\le i<j\le n}(b_j-b_i)(a_j-a_i)}{\prod_{1\le i,j\le n}(a_i+b_j)}$$

$$\Delta_6=\det\left(A+xJ\right)=\begin{vmatrix} a_{1,1}+x & a_{1,2}+x & \cdots & a_{1,n}+x \\ a_{2,1}+x & a_{2,2}+x & \cdots & a_{2,n}+x \\ \vdots & \vdots & & \vdots \\ a_{n,1}+x & a_{n,2}+x & \cdots & a_{n,n}+x \end{vmatrix}=\begin{vmatrix} a_{1,1}+x & a_{1,2}-a_{1,1} & \cdots & a_{1,n}-a_{1,1} \\ a_{2,1}+x & a_{2,2}-a_{2,1} & \cdots & a_{2,n}-a_{2,1} \\ \vdots & \vdots & & \vdots \\ a_{n,1}+x & a_{n,2}-a_{n,1} & \cdots & a_{n,n}-a_{n,1} \end{vmatrix}=f(x)=\alpha x+\det A$$

$$f(-x)=\det\left(A-xJ\right)=\det\left(A^{\mathsf{T}}-xJ^{\mathsf{T}}\right)=\det\left(-A-xJ\right)=(-1)^{2n}\det\left(A+xJ\right)=f(x)\quad .$$

$$\Rightarrow\quad \alpha=0\quad\Rightarrow\quad \boxed{\Delta_6=\det A}$$